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A Remark on the KAM Theorem Applied to a Four-Vortex System

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We consider a planar four-vortex system with unit intensities and apply the KAM theorem for two-dimensional tori with fixed frequency. We obtain a rigorous lower bound for the stochasticity threshold of the torus with rotation number $\omega = (\sqrt{5} - 1)/2$ and compare our result with numerical experiments.

KEY WORDS: KAM theory; quasiperiodic motions; four-vortex system; stochasticity threshold.

1. INTRODUCTION

We consider a planar four-vortex system with positive intensities $k_1,..., k_4$. As shown by Ziglin,[1] the system is not integrable and one can reduce it to a two-degree-of-freedom Hamiltonian system,[2,3] for which the corresponding Hamiltonian looks like a Hamiltonian near an integrable one:

$$H(A_1, A_2, \phi_1, \phi_2; \varepsilon) = h(A_1, A_2) + \varepsilon f(A_1, A_2, \phi_1, \phi_2; \varepsilon)$$

here the perturbing parameter ε is equal to the ratio between the average distance $(d_{12} + d_{34})/2$ $[d_{ij} = \text{dist}(i, j)]$ and the distance d of the mass centers of the couples $(1, 2)$ and $(3, 4)$ (see Fig. 1 below).

Khanin[2] applied the KAM theorem to the four-vortex system in order to state the existence of a nonempty set of initial conditions for which one has a quasiperiodic motion. In this paper we want to complete Khanin's proof giving an explicit rigorous lower bound on the size of the perturbation necessary for the disappearence of an analytic KAM torus with given rotation number ω. We fix the strengths equal to one and the pulsation $\omega = (\sqrt{5} - 1)/2$.

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An application of the KAM computer-assisted algorithm given in ref. 4 shows that for $|\varepsilon| < \varepsilon_{\mathrm{KAM}} \equiv 7.81 \times 10^{-23}$ there exists an invariant torus with rotation number ω (see Section 3).

Since a numerical integration of the equations of motion (see Section 4) shows that the critical value at which we expect the transition to a chaotic behavior is $\varepsilon_c \simeq 0.023$, then the ratio between the numerical and the KAM values is $\varepsilon_c/\varepsilon_{\mathrm{KAM}} \simeq 2.94 \times 10^{20}$, showing that the theoretical result is very far from reality. We used the complicated algorithm described in ref. 4, since a naive application of KAM theory would lead to an even worse estimate (say, $\varepsilon < 10^{-34}$).

We remark that one could apply some perturbation theory techniques,[5] or different KAM algorithms,[6] to obtain a better bound in simpler problems. But the complicate structure of the four-vortex Hamiltonian does not allow an easy application of these techniques.

2. HAMILTONIAN OF FOUR VORTICES

We consider a planar four-vortex system with positive strengths k_1, k_2, k_3, k_4; in Cartesian coordinates (x_i, y_i), $i = 1,..., 4$ (Fig. 1), the Hamiltonian takes the form

$$H = -\frac{1}{2\pi} \sum_{\substack{i,j=1 \\ i<j}}^{4} k_i k_j \ln d_{ij} \tag{2.1}$$

where $d_{i,j} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2}$.

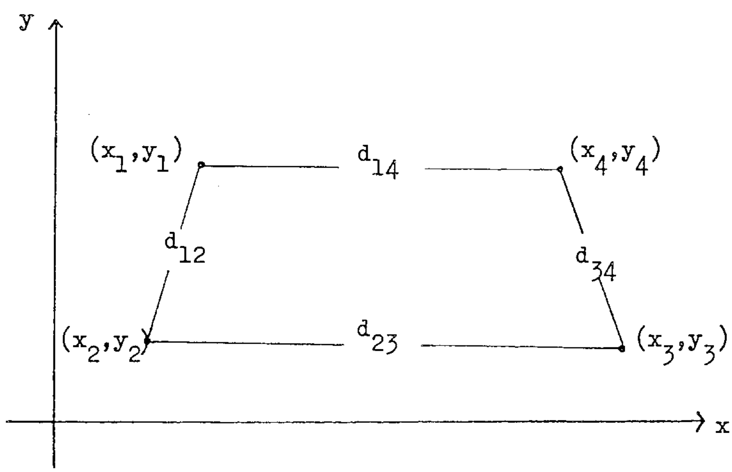

Fig. 1. Representation of the planar four-vortex system in cartesian coordinates.

It is well known that the system under consideration admits four different first integrals given by the Hamiltonian itself, the center of vorticity, and the angular momentum. Nevertheless, as shown by Ziglin,[1] these integrals are not sufficient to state the integrability of the four-vortex system.

To have the Hamiltonian (2.1) in action–angle variables, one has to make some changes of coordinates,[2] using the first integrals of the motion. After some computations, one obtains a two-degree-of-freedom Hamiltonian system whose Hamiltonian, expressed in terms of action–angle variables $(A_1, A_2, \phi_1, \phi_2)$ (and of a perturbing parameter ε, whose precise definition will be given below), takes the form

$$\begin{gathered} H(A_1, A_2, \phi_1, \phi_2; \varepsilon) = h(A_1, A_2) + g(A_1, A_2, \phi_1, \phi_2; \varepsilon) \\ (A_1, A_2) \in R^2, \qquad (\phi_1, \phi_2) \in T^2 \end{gathered} \tag{2.2}$$

($T^2 \equiv$ standard bidimensional torus), with $g(A_1, A_2, \phi_1, \phi_2; 0) = 0$.

The physical meaning of the (A, ϕ) variables is the following:

$$\left(\left[\frac{2(k_1+k_2)}{k_1 k_2} A_1\right]^{1/2}, \phi_1\right)$$

are the polar coordinates of the vector $(x_1 - x_2, y_1 - y_2)$; and

$$\left(\left[\frac{2(k_3+k_4)}{k_3 k_4} A_2\right]^{1/2}, \phi_2\right)$$

are the polar coordinates of the vector $(x_3 - x_4, y_3 - y_4)$.

In order to define the perturbing parameter ε, let us introduce the variables (A_3, ϕ_3) such that

$$\left(\left[\frac{2\sum_{i=1}^4 k_i}{(k_1+k_2)(k_3+k_4)} A_3\right]^{1/2}, \phi_3\right)$$

are the polar coordinates of the vector $(\chi_1 - \chi_2, \eta_1 - \eta_2)$, where

$$\chi_1 \equiv \frac{k_1 x_1 + k_2 x_2}{k_1 + k_2}, \qquad \chi_2 \equiv \frac{k_3 x_3 + k_4 x_4}{k_3 + k_4}$$

$$\eta_1 \equiv \frac{k_1 y_1 + k_2 y_2}{k_1 + k_2}, \qquad \eta_2 \equiv \frac{k_3 y_3 + k_4 y_4}{k_3 + k_4}$$

It turns out that the quantity $\lambda = A_1 + A_2 + A_3$ is a constant (since the Hamiltonian depends only upon $\phi_i - \phi_j$; see ref. 2). Thus we define ε as

$$\varepsilon \equiv \mu/\lambda, \qquad \text{with} \quad \mu \equiv (A_1^0 + A_2^0)/2$$

where A_1^0 and A_2^0 are the initial unperturbed values of A_1 and A_2 (namely, with $\varepsilon = 0$). Let us denote by d the distance between the centers of mass of the couples (1, 2) and (3, 4) (see Fig. 1). Then the perturbing parameter ε is in turn related to the ratio between the average distance $(d_{12} + d_{34})/2$ and d.

Now, if d is much bigger than the distances d_{12} and d_{34}, ε is very small and the system becomes near to an integrable model.

We report here the explicit expressions for the functions h and g, with the assumption of identical strengths: $k_1 = \cdots = k_4 = k$:

$$h(A_1, A_2) \equiv -\frac{k^2}{4\pi}(\ln A_1 + \ln A_2)$$

$$\begin{aligned}
&g(A_1, A_2, \phi_1, \phi_2; \varepsilon)\\
&\quad \equiv -\frac{k^2}{4\pi}\Bigg[\ln\Bigg(1 + \frac{\varepsilon}{2\mu}\Bigg\{(1-2k)A_1 + (1-2k)A_2\\
&\qquad + 2\left[2A_1\left(\frac{\mu}{\varepsilon} - A_1 - A_2\right)\right]^{1/2}\cos\phi_1\\
&\qquad - 2\left[2A_2\left(\frac{\mu}{\varepsilon} - A_1 - A_2\right)\right]^{1/2}\cos\phi_2 - 2(A_1A_2)^{1/2}\cos(\phi_1 - \phi_2)\Bigg\}\Bigg)\\
&\qquad + \ln\Bigg(1 + \frac{\varepsilon}{2\mu}\Bigg\{(1-2k)A_1 + (1-2k)A_2\\
&\qquad + 2\left[2A_1\left(\frac{\mu}{\varepsilon} - A_1 - A_2\right)\right]^{1/2}\cos\phi_1\\
&\qquad + 2\left[2A_2\left(\frac{\mu}{\varepsilon} - A_1 - A_2\right)\right]^{1/2}\cos\phi_2 + 2(A_1A_2)^{1/2}\cos(\phi_1 - \phi_2)\Bigg\}\Bigg)\\
&\qquad + \ln\Bigg(1 + \frac{\varepsilon}{2\mu}\Bigg\{(1-2k)A_1 + (1-2k)A_2\\
&\qquad - 2\left[2A_1\left(\frac{\mu}{\varepsilon} - A_1 - A_2\right)\right]^{1/2}\cos\phi_1\\
&\qquad - 2\left[2A_2\left(\frac{\mu}{\varepsilon} - A_1 - A_2\right)\right]^{1/2}\cos\phi_2 + 2(A_1A_2)^{1/2}\cos(\phi_1 - \phi_2)\Bigg\}\Bigg)\\
&\qquad + \ln\Bigg(1 + \frac{\varepsilon}{2\mu}\Bigg\{(1-2k)A_1 + (1-2k)A_2\\
&\qquad - 2\left[2A_1\left(\frac{\mu}{\varepsilon} - A_1 - A_2\right)\right]^{1/2}\cos\phi_1\\
&\qquad + 2\left[2A_2\left(\frac{\mu}{\varepsilon} - A_1 - A_2\right)\right]^{1/2}\cos\phi_2 - 2(A_1A_2)^{1/2}\cos(\phi_1 - \phi_2)\Bigg\}\Bigg)\Bigg]
\end{aligned} \tag{2.3}$$

3. KAM ESTIMATES

In order to apply the analytic version of the KAM theorem in the formulation given in ref. 4, we want to have an Hamiltonian of the form

$$H(A_1, A_2, \phi_1, \phi_2; \varepsilon) = h(A_1, A_2) + \varepsilon f(A_1, A_2, \phi_1, \phi_2; \varepsilon) \tag{3.1}$$

To this end, we expand the logarithms in (2.3) in Taylor series with respect to ε.

Once (2.2) is reduced to the form (3.1), we fix the particular torus $T(\omega) \equiv \{A_0\} \times T^2$, with

$$\omega \equiv \left(\frac{\partial h(A_1^0, A_2^0)}{\partial A_1}, \frac{\partial h(A_1^0, A_2^0)}{\partial A_2} \right) = \left(\frac{\sqrt{5}-1}{2}, 1 \right) \tag{3.2}$$

Our aim is to investigate the stability of $T(\omega)$, giving an explicit estimate of the critical size ε_c at which we expect the disappearance of such a torus.

Before stating the theorem, let us introduce some definitions; set

$$S_R(A_0) = \{A \in R^2 / |A - A_0| \leqslant R\}$$

$$\hat{S}_R(A_0) = \{A \in C^2 / |A - A_0| \leqslant R\}$$

$$D_\xi^2 = \{z \in C^2 / e^{-\xi} < |z_i| < e^{\xi}, i = 1, 2\}$$

Note that z is related to ϕ by $z = (z_1, z_2) = (e^{i\phi_1}, e^{i\phi_2})$.

To apply the KAM theorem, one has to satisfy the following conditions:

(i) h and f must be holomorphic in $\hat{S}_R(A_0)$, $\hat{S}_R(A_0) \times D_\xi^2$.

(ii) ω must satisfy the Diophantine inequality:

$$|\omega \cdot \nu|^{-1} \leqslant C\,|\nu|^2, \qquad \forall \nu \in Z^2 \text{ and for some } C > 0$$

(iii) h has to be nondegenerate:

$$\det \left[\frac{\partial^2 h}{\partial A^2}(A) \right] \neq 0, \qquad \forall A \in \hat{S}_R(A_0)$$

In our problem (i) is satisfied with the choice $R = 0.07$, $\xi = 3$; (ii) is fulfilled by the pulsation (3.2) with a constant $C = (3 + \sqrt{5})/2$; and (iii) is automatically satisfied.

Then we have the following result:

Theorem. Consider the Hamiltonian (3.1) corresponding to a planar four-vortex system with equal strengths and fix a torus $T(\omega)$ with rotation number (3.2). Set $R = 0.07$ and $\xi = 3$; then, for every $|\varepsilon| < 7.81 \times 10^{-23}$ there exists in $S_R(A_0)$ a torus $T_\varepsilon(\omega)$ with rotation number ω, which is an invariant torus for the Hamiltonian flow associated with (3.1).

The proof of the theorem follows by an easy generalization of the computer-assisted KAM algorithm given in ref. 4 to a two-dimensional problem. In the KAM proof one performs a canonical change of variables in order to reduce the initial Hamiltonian $H_0 = h_0 + \varepsilon f_0$ to one of the form $H_1 = h_1 + \varepsilon^2 f_1$. Then, one proceeds by a superconvergent iteration of this method, reducing $H_j = h_j + \varepsilon^{2^j} f_j$ to the Hamiltonian $H_{j+1} = h_{j+1} + \varepsilon^{2^{j+1}} f_{j+1}$.

In order to be able to make such a reduction, one has to satisfy a smallness requirement on ε at each step $H_j \to H_{j+1}$ for every $j \geq 0$. These smallness conditions are "optimized" in ref. 4, to which we refer for the details of the proof. Moreover, since these conditions depend crucially on the choice of the analyticity parameters R, ξ, we set $R = 0.07$, $\xi = 3$ in order to "optimize" the final result. Notice that in our computations we have set $k_1 = \cdots = k_4 = 1$, since a rescaling of the strengths by a same factor does not affect the dynamics of the system.

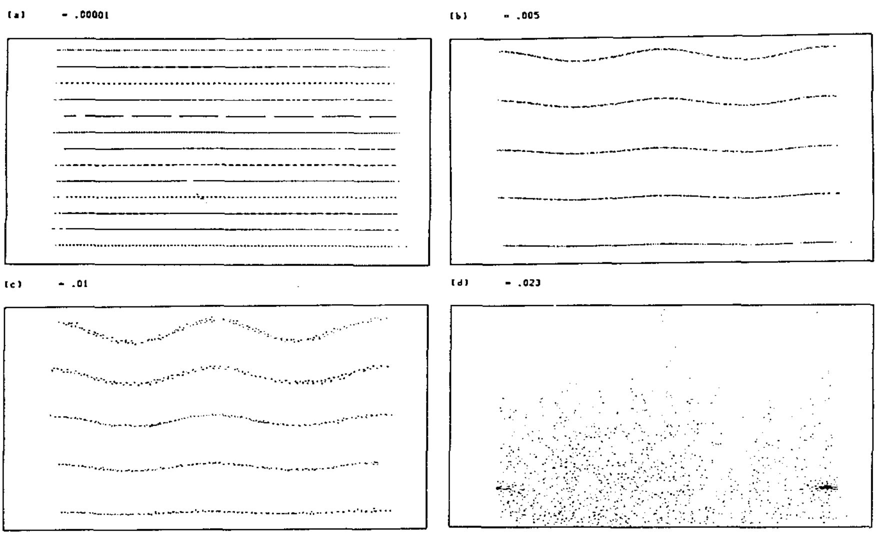

Fig. 2. Poincaré maps on the (ϕ_1, A_1) plane $(-\pi \leq \phi_1 \leq \pi,\ 0.07 \leq A_1 \leq 0.2)$ for different initial values of the perturbing parameter ε. (a) refers to $\varepsilon = 0.00001$, (b) $\varepsilon = 0.005$, (c) $\varepsilon = 0.01$, (d) $\varepsilon = 0.023$.

4. NUMERICAL EXPERIMENTS

We compare the theoretical result given in Section 3 with a naive numerical experiment. Starting with the Hamiltonian (2.2), we perform a direct numerical integration of the Hamilton equations, using the Runge-Kutta method at the fourth order.

A Poincaré map on the (ϕ_1, A_1) plane shows that for $\varepsilon \simeq 0$ the torus $T(\omega)$ is described by an almost straight line (see Fig. 2a). Increasing ε, the line is gradually distorted (see Figs. 2b and 2c), until one reaches the critical value ε_c, at which chaos appears (see Fig. 2d).

The numerical result for the stochasticity threshold is $\varepsilon_c = 0.023$, which is in accordance with the critical value given by Aref and Pomphrey.(7)

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